P-spectrum and collapsing of connected sums, calculus of the limit

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p-SPECTRUM AND COLLAPSING OF CONNECTED SUMS
(P-SPECTRE ET EFFONDREMENT DE SOMMES CONNEXES)

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ABSTRACT. The goal of the present paper is to calculate the limit of spectrum of the Hodge-de Rham operator under the perturbation of collapse of one part of a connected sum. It takes place in the general problem of blowing up conical singularities introduced in [Maz06] and [Row08].

Le but de ce travail est de calculer la limite du spectre de l'opéateur de Hodge-de Rham dans la perturbation obtenue par effondrement d'une moitié d'une somme connexe. Ce problème rentre dans le cadre de l'éclatement des singularités coniques introduites dans [Maz06] et [Row08].

1. INTRODUCTION

It is a common problem in differential geometry to study the limit of spectrum of Laplace type operators under singular perturbations of the metrics, especially for the Hodge-de Rham operator acting on differential forms. The first reason is the topological meaning of this operator and the fact that by singular perturbations of the metric one can change the topology of the manifold. Among a lot of works on this subject we must recall the study of the adiabatic limit, started by Mazzeo and Melrose in [MM90] and developped by many authors, we can mention also the collapse of thin handles started in [AC95] and accomplished in [ACP07].

The singular perturbation we study here takes place in the general framework of resolution blowups presented in [Maz06], our situation is the collapse of one part in a connected sum, which is explained by the following figure.

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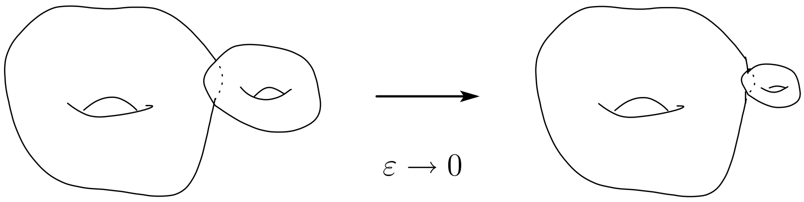

FIGURE 1. collapsing of M_ε

More precisely, if the manifold M, of dimension m, is the connected sum of M_1 and M_2 around the common point p_0, endowed with Riemannian metrics g_1, g_2, then, for the collapse of one part of the connected sum we study the dependence on $\varepsilon \to 0$ for the manifold

$$M_\varepsilon := (M_1 - B(p_0, \varepsilon)) \cup \varepsilon.(M_2 - B(p_0, 1)),$$

where $\varepsilon.(M_2 - B(p_0, 1))$ means $(M_2 - B(p_0, 1), \varepsilon^2 g_2)$.

To make this construction clear, we can suppose that the two metrics are flat around the point p_0, then the boundaries of $(M_1 - B(p_0, \varepsilon), g_1)$ and $(M_2 - B(p_0, 1), \varepsilon^2 g_2)$ are isometric, and can be identified. One can then define geometrically M_ε as a Riemannian manifold C^∞ by part.

In the terminology of [Maz06], M_ε is the resolution blowup of the (singular) space M_1 with $\mathbb{S}^n$ as *link* and $\widetilde{M_2}$ as *asymptotically conical manifold*, if $\widetilde{M_2}$ is the complete manifold obtained by gluing of the exterior of a ball in the Euclidean space to the boundary of $M_2 - B(p_0, 1)$. In fact, Rowlett studies, in [Row08], the convergence of the spectrum of generalized Laplacian on such a situation of blowup of one isolated conical singularity (Mazzeo presents more genral sinbularities in [Maz06]). Her result gives the convergence of the spectrum to the spectrum of an operator on M_1, it requires an hypothesis on $\widetilde{M_2}$. Our result is less general, applied only to the case of the Hodge-de Rham operator, but it does not require this hypothesis and the limit spectrum takes care of $\widetilde{M_2}$, see Theorem C below.

Maybe, the more important interest of our study is that we introduce new techniques: to solve this kind of problems, we have to identify a good elliptic limit problem, this means for the M_2 part a good boundary problem on $M_2 - B(p_0, 1)$ at the limit. It appears that, on difference with the problem of thin handles in [AC95] or the connected sum problem studied in [T02] for functions, this boundary problem is not a kind of local but 'global': we have to introduce a condition of the Atiyah-Patodi-Singer (APS) type, as defined in [APS75].

Indeed, these APS boundary conditions are related to the Fredholm theory on the link $\widetilde{M_2}$, as explained by Carron in [C01], details are given below.

We shall study the more general blowup of conical spaces in the future.

1.1. **The results.** As mentioned above, the manifold M, of dimension $m \geq 3$ (there is no problem in dimension 2), is the connected sum of two Riemannian manifolds (M_1, g_1) and (M_2, g_2) around the common point p_0, and we suppose that the metrics are such that the boundaries of $(M_1 - B(p_0, \varepsilon), g_1)$ and $(M_2 - B(p_0, 1), \varepsilon^2 g_2)$ are isometric for all ε small enough. As a consequence, (M_1, g_1) is flat in a neighborhood of p_0 and $\partial(M_2 - B(p_0, 1))$ is the standard sphere. Indeed one can write $g_1 = dr^2 + r^2 h(r)$ in the polar coordinates around $p_0 \in M_1$ and the metric $h(r)$ on the sphere converges, as $r \to 0$, to the standard metric. But if the boundaries of $(M_1 - B(p_0, \varepsilon), g_1)$ and $(M_2 - B(p_0, 1), \varepsilon^2 g_2)$ are isometric for all ε small enough, then $h(r)$ is constant for r small enough, the conclusion follows.

One can then define geometrically $M_\varepsilon := (M_1 - B(p_0, \varepsilon)) \cup \varepsilon.(M_2 - B(p_0, 1))$ as the connected sum obtained by the collapse of M_2 (the question of the metric on M_ε is discussed below). On such a manifold, the Gauß-Bonnet operator D_ε, Sobolev spaces and also the Hodge-de Rham operator Δ_ε can be defined as follows (the details are given in [AC95]): on a manifold $X = X_1 \cup X_2$, which is the union of two Riemannian manifolds with isometric boundaries, if D_1 and D_2 are the Gauß-Bonnet "$d + d^*$" operators acting on the differential forms of each part, the quadratic form

$$q(\varphi) = \int_{X_1} |D_1(\varphi \restriction_{X_1})|^2 \, d\mu_{X_1} + \int_{X_2} |D_2(\varphi \restriction_{X_2})|^2 \, d\mu_{X_2}$$

is well-defined and closed on the domain

$$\mathrm{Dom}(q) := \{\varphi = (\varphi_1, \varphi_2) \in H^1(\Lambda T^* X_1) \times H^1(\Lambda T^* X_2) \mid \varphi_1 \restriction_{\partial X_1} \overset{L_2}{=} \varphi_2 \restriction_{\partial X_2}\},$$

where the boundary values $\varphi_i \restriction_{\partial X_i}$ are considered in the sense of the trace operator, and on this space the total Gauß-Bonnet operator $D(\varphi) = (D_1(\varphi_1), D_2(\varphi_2))$ is defined and selfadjoint. For this definition, we have, in particular, to identify $(\Lambda T^* X_1) \restriction_{\partial X_1}$ and $(\Lambda T^* X_2) \restriction_{\partial X_2}$. This can be done by decomposing the forms in tangential and normal parts (with inner normal), the equality above means then that the tangential parts are equal and the normal parts opposite. This definition generalizes the definition in the smooth case.

The Hodge-de Rham operator $(d + d^*)^2$ of X is then defined as the operator obtained by the polarization of the quadratic form q. This gives compatibility conditions between φ_1 and φ_2 on the common boundary. We do not give details on these facts because, as remarked in the next section, it is sufficient to work with smooth metrics on M.

The multiplicity of 0 in the spectrum of Δ_ε is given by the cohomology, it is then independent of ε and can be related to the cohomology of each part by the Mayer-Vietoris argument. The point is to study the convergence of the other eigenvalues, the so-called *positive spectrum*, as $\varepsilon \to 0$. The second author has shown in [T03], Theorem 4.4, p.21, a result of boundedness

Proposition A (Takahashi). *The superior limit of the k-th* positive *eigenvalue on p-forms of M_ε is bounded, as $\varepsilon \to 0$, by the k-th* positive *eigenvalue on p-forms of M_1.*

We show here that it is also true for the lower bound. Let φ_ε be a family of eigenforms on M_ε of degree p for the Hodge-de Rham operator:

$$\Delta_\varepsilon \varphi_\varepsilon = \lambda^p(M_\varepsilon)\varphi_\varepsilon \; ; \; \lim_{\varepsilon \to 0} \lambda^p(M_\varepsilon) = \lambda^p < +\infty.$$

Proposition B. *If $\lambda^p(M_\varepsilon) \neq 0$, then $\lambda^p \neq 0$ and λ^p belongs to the spectrum of the Hodge-de Rham operator on (M_1, g_1).*

The first point is a consequence of the application of the so-called McGowan's lemma; indeed M_ε has no small eigenvalues as is shown in Proposition 1 below. To prove the convergent part of the proposition, we shall decompose the eigenforms using the good control of the APS-boundary term. More precisely, there exists an elliptic extension $\mathcal{D}_2$ of the Gauß-Bonnet operator D_2 on $M_2(1) = M_2 - B(p_0, 1)$ and a family ψ_ε bounded in $H^1(M_1) \times \mathrm{Dom}(\mathcal{D}_2)$ such that $\|\varphi_\varepsilon - \psi_\varepsilon\| \to 0$ with ε.

This extension is defined by *global* boundary conditions, the conditions of APS type, in relation with the works of Carron about operators non parabolic at infinity developped in [C01], see proposition 5.

If we make this construction for an orthonormal family of the k first eigenforms, we obtain, with the help of Proposition A, our main theorem

Theorem C. *Let $M_\varepsilon = (M_1 - B(p_0, \varepsilon)) \cup \varepsilon.(M_2 - B(p_0, 1))$ be the connected sum of the two Riemannian manifolds M_1 and $\varepsilon.M_2$ of dimension $m = n + 1$. For $p \in \{1, \ldots, n\}$, let $0 < \lambda_1^p(M_1) \leq \lambda_2^p(M_1), \ldots$ be the* positive *spectrum of the Hodge-de Rham operator on the p-forms of M_1 and $0 < \lambda_1^p(M_\varepsilon) \leq \lambda_2^p(M_\varepsilon), \ldots$ the* positive *spectrum of the Hodge-de Rham operator on the p-forms of M_ε. Then, for all $k \geq 1$ we obtain*

$$\lim_{\varepsilon \to 0} \lambda_k^p(M_\varepsilon) = \lambda_k^p(M_1).$$

Moreover, the multiplicity of 0 is given by the cohomology and

$$H^p(M_\varepsilon; \mathbb{R}) \cong H^p(M_1; \mathbb{R}) \oplus H^p(M_2; \mathbb{R}).$$

Remark 1. A. *The result of convergence of the positive spectrum is also true for $p = 0$ and has been shown in* [T02]. *Naturally $H^0(M_\varepsilon; \mathbb{R}) \cong H^0(M_1; \mathbb{R}) = \mathbb{R}$. By the Hodge duality this solves also the case $p = m$.*

1.2. **Applications.** Results on spectral convergence in singular situations can be used to give examples or counter examples, concerning possible links between spectral and geometric properties. For instance, Colbois and El Soufi have introduced in [CE03] the notion of *conformal spectrum* as the supremum, for each integer k, of the value of the k-th eigenvalue on a conformal class of metrics with fixed volume.

Using the result of [T02], they could show that the conformal spectrum of a compact manifold is always bounded from below by the conformal spectrum of the standard sphere of the same dimension.

In the same way, applying the Theorem C to the case $M_1 = \mathbb{S}^m$ and $M_2 = M$, we obtain

Corollary D. *Let (M, g) be a compact Riemannian manifold of dimension m, for any degree p, any integer $N \geq 1$ and any $\varepsilon > 0$, there exists on M a metric $\overline{g}$ conformal to g such that the N first positive eigenvalues on the p-forms are ε-close to the N first positive eigenvalues on the p-forms of the standard sphere with the same dimension and the same volume as (M, g).*

Remark 1.B. *For the completion of the panorama on this subject, let us recall that Jammes has shown, in* [J07]*, that in dimension $m \geq 4$ the infimum of the p-spectrum in a conformal class, with fixed volume, is 0 for $2 \leq p \leq m-2$ and $p \neq \frac{m}{2}$ but has a positive lower bound for $p = \frac{m}{2}$.*

Another example is the *Prescription of the spectrum.* This question was introduced by Colin de Verdière in [CdV86, CdV87] where he shows that he can impose any finite part of the spectrum of the Laplace-Beltrami operator on certain manifolds. To this goal, he introduced a very powerful technique of transversality, and shows that this hypothesis is satisfied on certain graphs and on certain manifolds [CdV88]. The other necessary argument is a theorem of convergence. The solution of the problem of prescription, with limitation concerning multiplicity, has been given by Guerini in [G04] for the Hodge-de Rham operator, and Jammes has proved a result of prescription, without multiplicity, in a conformal class of the metric in [J08a], for certain degrees of the differential forms, what is compatible with the retricted result mentioned above. In this context, our result gives, for example,

Corollary E. *Let g_0 be a metric on the sphere of dimension m. If g_0 satisfies the Strong Arnol'd Hypothesis, following the terminology of* [CdV86]*, for the eigenvalue $\lambda \neq 0$ on differential forms of degree p on the sphere, then for any closed manifold M, there exists a metric such that λ belongs to the spectrum of the Hodge-de Rham operator on p-forms with the same multiplicity.*

Indeed, we take a metric g_2 on M, and for any metric g_1 close to g_0, the positive spectrum of $M_\varepsilon = \mathbb{S}^m \# \varepsilon.M$ converges, as $\varepsilon \to 0$ to the spectrum of $\mathbb{S}^m$. Then, the Strong Arnol'd Hypothesis assures that the map which associates to g_1 the *spectral quadratic form* relative to a small interval I around λ has also, for ε small enough, the matrix $\lambda \cdot \mathrm{Id}$ in its image.

Here, by spectral quadratic form, we mean the quadratic form defined by the Hodge-de Rham operator, restricted to the eigenspace of eigenforms with eigenvalues in I. To consider this space as a space of matrix, we have to construct small isometries between the different eigenspaces, the details are in [CdV88].

This result could be used to prescribe high multiplicity for the spectrum of the Hodge-de Rham operator. Recall that Jammes had obtained partial results on this subject in [J08b], his work is based on a convergence theorem (theorem 2.8) where the limit is the Hodge-de Rham operator with absolute boundary condition on a domain, he uses also the fact that the Strong Arnol'd Hypothesis is satisfied on spheres of dimension 2, as proved in [CdV88]. It would be interesting to obtain such a result on spheres of bigger dimension, the result of [CdV88] uses the conformal invariance specific to this dimension.

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We now proceed to prove the theorems. Let us first describe the metrics precisely.

2. Choice of the metric

From now on, we denote

$$M_2(1) := M_2 - B(p_0, 1).$$

It is supposed here that the ball $B(p_0, 1)$ can really be embedded in the manifold M_2, this can always be satisfied by a scaling of the metric g_2 on M_2.

Recall that Dodziuk has proved in [D82, Prop. 3.3] that if two metrics g, $\overline{g}$ on the same compact manifold satisfy

$$\mathrm{e}^{-\eta} g \le \overline{g} \le \mathrm{e}^{\eta} g. \tag{2.1}$$

Then, the corresponding eigenvalues of the Hodge-de Rham operator acting on p-forms satisfy

$$\mathrm{e}^{-(n+2p)\eta} \lambda_k^p(g) \le \lambda_k^p(\overline{g}) \le \mathrm{e}^{(n+2p)\eta} \lambda_k^p(g).$$

This result is based on the fact that the multiplicity of 0 is given by the cohomology and the positive spectrum by exact forms, hence the min-max formula does not involve derivatives of the metric; it stays valid if one of the two metrics is only smooth by part, because in the last case the Hodge decomposition still holds true.

Then, for a metric g_1 on M_1 there exists, for each $\eta > 0$ a metric $\overline{g}_1$ on M_1 which is flat on a ball B_η centered at p_0 and such that

$$\mathrm{e}^{-\eta} g_1 \le \overline{g}_1 \le \mathrm{e}^{\eta} g_1.$$

Then our result can be extended to any other construction which does not suppose that the metric g_1 is flat in a neighborhood of p_0.

Now, we regard M_ε as the union of $M_1 - B(p_0, 3\varepsilon)$ and $\varepsilon.\overline{M}_2(1)$, where $\overline{M}_2(1) = \big(B_{\mathbb{R}^m}(0,3) - B_{\mathbb{R}^m}(0,1)\big) \cup M_2(1)$ is endowed with a metric only smooth by part: the Euclidean metric on the first part and the restriction of g_2 on the second part. But this metric can be approached, as close as we want, by a smooth metric which is still flat on $B_{\mathbb{R}^m}(0,3) - B_{\mathbb{R}^m}(0,\frac{3}{2})$ and these two metrics will satisfy the estimate (2.1). Thus, replacing 3ε by ε for simplicity, we can suppose, without loss of generality, that we are in the following situation:

The manifold $M_2(1)$ is endowed with a metric which is conical (flat) near the boundary, namely $g_2 = ds^2 + (1-s)^2 h$, h being the canonical metric of the sphere $\mathbb{S}^n = \partial(M_2(1))$, and $s \in [0,\frac{1}{2})$ being the distance from the boundary ($M_2(1)$ looks like a trumpet) and $M_1(\varepsilon) = M_1 - B(p_0,\varepsilon)$ with a conical metric $g_1 = dr^2 + r^2 h$ around the point p_0. Thus, $M_\varepsilon = M_1(\varepsilon) \cup \varepsilon.M_2(1)$ is a smooth Riemannian manifold.

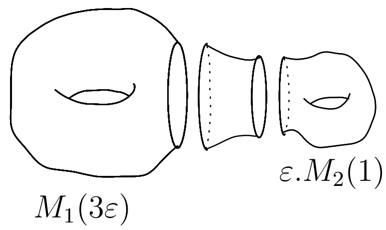

FIGURE 2. smoothing of $(M_\varepsilon, g_\varepsilon)$

Let $\mathcal{C}_{a,b}$ be the cone $(a,b)\times\mathbb{S}^n$ endowed with the (conical) metric $dr^2 + r^2 h$.

3. Small eigenvalues

Let's show that $M(\varepsilon)$ has no small eigenvalues.

Proposition 1. *If $1 \le p \le n$, There is a constant $\lambda_0 > 0$ such that, if $1 \le p \le n$,*

$$\lambda_\varepsilon \neq 0 \Rightarrow \lambda_\varepsilon \ge \lambda_0.$$

Proof. We shall use the McGowan's lemma as enonciated in [GP95]. Recall that this lemma, in the spirit of Mayer Vietoris theorem, gives control of positive eigenvalues in terms of positive eigenvalues of certain covers with certain boundary conditions. We use the cover $M_\varepsilon = M_1(\varepsilon) \cup \varepsilon.(M_2(1) \cup \mathcal{C}_{1,2})$. Let

$$U_1 = M_1(\varepsilon) \text{ and } U_2 = \varepsilon.(M_2(1) \cup \mathcal{C}_{1,2})$$

then $U_{1,2} = U_1 \cap U_2 = \varepsilon.\mathcal{C}_{1,2}$ and $H^{p-1}(U_1 \cap U_2) = 0$ for $1 < p \le n$.

The lemma 1 of [GP95] asserts that, in this case and for these values of p, the first positive eigenvalue of the Hodge-Laplace operator on exact p-forms of M_ε is, up to a power of 2, bounded from below by

$$\lambda_0(\varepsilon) = \Big((\frac{1}{\mu^p(U_1)} + \frac{1}{\mu^p(U_2)})(\frac{\omega_{p,m} c_\rho}{\mu^{p-1}(U_{1,2})} + 1) \Big)^{-1}$$

where $\mu^k(U)$ is the first positive eigenvalue of the Laplacian acting on exact k-forms of U and satisfying absolute boundary conditions, $\omega_{p,m}$ is a combinatorial constant and c_ρ is the square of an upper bound of the first derivative of a partition of 1 subordinate to the cover.

For us $c_\rho, \mu^p(U_2)$ and $\mu^{p-1}(U_{1,2})$ are all of order ε^{-2}, but $\mu^p(U_1)$ is bounded for $p \le n$ as was shown in [AC93] (remark that the small eigenvalue exibited here in degree $m-1$ is in the coexact spectrum). This give a uniform bound for the exact spectrum of degree p with $1<p\leq n$ but the exact spectrum for 1-forms comes from the spectrum on function which has been studied in [T02], thus the exact spectrum is controled for $1\leq p\leq n$, by Hodge duality it gives a control for all the positive spectrum in these degrees. Finally we can assert that there exists $\lambda_0>0$ such that $\forall\varepsilon$, $\lambda_0(\varepsilon)>\lambda_0$. □

The proof of the main Proposition B needs some useful notations and estimates, it is the goal of the following section.

4. Estimates and tools

As in [ACP07] we use the following change of variables : with

$$\varphi_{\varepsilon|M_1(\varepsilon)}=\varphi_{1,\varepsilon} \ \text{ and } \ \varphi_{\varepsilon|M_2(1)}=\varepsilon^{p-m/2}\varphi_{2,\varepsilon}$$

we write on the cone

$$\varphi_{1,\varepsilon}=dr\wedge r^{-(n/2-p+1)}\beta_{1,\varepsilon}+r^{-(n/2-p)}\alpha_{1,\varepsilon}$$

and define $\sigma_1=(\beta_1,\alpha_1)=U(\varphi_1)$.

On the other part, it is more convenient to define $r=1-s$ for $s\in[0,1/2]$ and write $\varphi_{2,\varepsilon}=(dr\wedge r^{-(n/2-p+1)}\beta_{2,\varepsilon}+r^{-(n/2-p)}\alpha_{2,\varepsilon})$ near the boundary. Then we can define, for $r\in[1/2,1]$ (the boundary of $M_2(1)$ corresponds to $r=1$)

$$\sigma_2(r)=(\beta_2(r),\alpha_2(r))=U(\varphi_2).$$

The L_2 norm, for a form supported on M_1 in the cone $\mathcal{C}_{\varepsilon,1}$, has the expression

$$\|\varphi\|^2=\int_{M_1}|\sigma_1|^2dr\wedge d\mathrm{vol}_{\mathbb{S}^n}+\int_{M_2}|\varphi_2|^2d\mathrm{vol}_{M_2}$$

and the quadratic form on study is

$$q(\varphi)=\int_{M(\varepsilon)}|(d+d^*)\varphi|^2=\int_{M_1(\varepsilon)}|UD_1U^*(\sigma_1)|^2+\frac{1}{\varepsilon^2}\int_{M_2(1)}|D_2(\varphi_2)|^2 \tag{4.1}$$

where D_1, resp. D_2, are the Gauß-Bonnet operator of M_1, resp. M_2, namely $D_j=d+d^*$ acting on differential forms. In terms of σ_1, which, a priori, belongs to $C^\infty([\varepsilon,1[,C^\infty(\Lambda^{p-1}T^*\mathbb{S}^n)\oplus C^\infty(\Lambda^pT^*\mathbb{S}^n))$ the operator has, on the cone of M_1, the expression

$$UD_1U^*=\begin{pmatrix}0&1\\-1&0\end{pmatrix}\left(\partial_r+\frac{1}{r}A\right) \ \text{ with } \ A=\begin{pmatrix}\frac{n}{2}-P&-D_0\\-D_0&P-\frac{n}{2}\end{pmatrix}$$

where P is the operator of degree which multiplies by p a p-form, and D_0 is the Gauß-Bonnet operator of the sphere $\mathbb{S}^n$.

While the Hodge-deRham operator has, in these coordinates, the expression

$$U\Delta_1U^*=-\partial_r^2+\frac{1}{r^2}A(A+1). \tag{4.2}$$

The same expressions are valid for UD_2U^* and $U\Delta_2U^*$ near the boundary of $M_2(1)$ but we shall not use them because we need global estimates on this part.

The compatibility condition is, for the quadratic form, $\varepsilon^{1/2}\alpha_1(\varepsilon) = \alpha_2(1)$ and $\varepsilon^{1/2}\beta_1 = \beta_2(1)$ or

$$\sigma_2(1) = \varepsilon^{1/2}\sigma_1(\varepsilon). \tag{4.3}$$

The compatibility condition for the Hodge-deRham operator, of first order, is obtained by expressing that $D\varphi \sim (UD_1U^*\sigma_1, \frac{1}{\varepsilon}UD_2U^*\sigma_2)$ belongs to the domain of D. In terms of σ it gives

$$\sigma_2'(1) = \varepsilon^{3/2}\sigma_1'(\varepsilon). \tag{4.4}$$

Let ξ_1 be a cut-off function on M_1 around p_0:

$$0 \le r \le 1/2 \Rightarrow \xi_1(r) = 1 \text{ and } r \ge 1 \Rightarrow \xi_1(r) = 0.$$

Proposition 2. *For our given family φ_ε satisfying $\Delta(\varphi_\varepsilon) = \lambda_\varepsilon\varphi_\varepsilon$ with λ_ε bounded, the family $(1-\xi_1).\varphi_{1,\varepsilon}$ is bounded in $H^1(M_1)$.*

Then it remains to study $\xi_1.\varphi_{1,\varepsilon}$ which can be expressed with the polar coordinates, this is the goal of the next section.

Remark 3. *The same cannot be done with the componant on M_2 or more precisely this does not give what we want to prove, namely that this componant goes to 0 with ε. To do so we have first to concider $\varphi_{2,\varepsilon}$ in the domain of an elliptic operator, this is the main difficulty, in contrast with the case concerning functions. In fact we will decompose $\varphi_{2,\varepsilon}$ in a part which clearly goes to 0 and an other part which belongs to the domain of an elliptic operator, this operator is naturally D_2 but the point is to determine the boundary conditions.*

4.1. **Expression of the quadratic form.** — For any φ such that the componant φ_1 is supported in the cone $\mathcal{C}_{1,\varepsilon}$, one has, with $\sigma_1 = U\varphi_1$ and by the same calculus as in [ACP07] :

$$\begin{aligned}
\int_{\mathcal{C}_{\varepsilon,1}} |D_1\varphi|^2 d\mathrm{vol}_{g_\varepsilon} &= \int_\varepsilon^1 \left|\Big(\partial_r + \frac{1}{r}A\Big)\sigma_1\right|^2 dr \\
&= \int_\varepsilon^1 \Big[\, |\sigma_1'|^2 + \frac{2}{r}\langle\sigma_1', A\sigma_1\rangle + \frac{1}{r^2}|A\sigma_1|^2 \Big] dr \\
&= \int_\varepsilon^1 \Big[\, |\sigma_1'|^2 + \partial_r\Big(\frac{1}{r}\langle\sigma_1, A\sigma_1\rangle\Big) + \frac{1}{r^2}\big(\langle\sigma_1, A\sigma_r\rangle + |A\sigma_1|^2\big) \Big] dr \\
&= \int_\varepsilon^1 \Big[\, |\sigma_1'|^2 + \frac{1}{r^2}\langle\sigma_1, (A+A^2)\sigma_1\rangle \Big] dr - \frac{1}{\varepsilon}\big\langle\sigma_1(\varepsilon), A\sigma_1(\varepsilon)\big\rangle.
\end{aligned}$$

we then have

$$q(\varphi)=\int_\varepsilon^1\Big[\,|\sigma_1'|^2+\frac{1}{r^2}\langle\sigma_1,(A+A^2)\sigma_1\rangle\,\Big]dr-\frac{1}{\varepsilon}\big\langle\sigma_1(\varepsilon),A\sigma_1(\varepsilon)\big\rangle$$
$$+\frac{1}{\varepsilon^2}\int_{M_2(1)}|D_2\varphi_2|^2 \quad (4.5)$$

On the other hand we have, as well,

$$\begin{aligned}\int_{\mathcal{C}_{1/2,1}}|D_2\varphi|^2 d\mathrm{vol}_{g_\varepsilon}&=\int_{1/2}^1\left|\Big(\partial_r+\frac{1}{r}A\Big)\sigma_2\right|^2dr\\&=\int_{1/2}^1\Big[\,|\sigma_2'|^2+\frac{1}{r^2}\langle\sigma_2,(A+A^2)\sigma_2\rangle\,\Big]dr\\&\quad+\big\langle\sigma_2(1),A\sigma_2(1)\big\rangle-\big\langle\sigma_2(1/2),A\sigma_2(1/2)\big\rangle.\end{aligned}$$

Thus the first boundary terms annihilate, and one has also

$$q(\varphi)=\int_\varepsilon^1\Big[\,|\sigma_1'|^2+\frac{1}{r^2}\langle\sigma_1,(A+A^2)\sigma_1\rangle\,\Big]dr+$$
$$\frac{1}{\varepsilon^2}\int_{1/2}^1\Big[\,|\sigma_2'|^2+\frac{1}{r^2}\langle\sigma_2,(A+A^2)\sigma_2\rangle\,\Big]dr-\frac{1}{\varepsilon^2}\big\langle\sigma_2(1/2),A\sigma_2(1/2)\big\rangle. \quad (4.6)$$

We remark that the boundary term $-\big\langle\sigma_2(1/2),A\sigma_2(1/2)\big\rangle$ is positif if σ_2 belongs to the eigenspace of A with negative eigenvalues. In fact we know the spectrum of A:

4.2. **Spectrum of** A. — It has been calculated in [BS88]. By their result, we have that the spectrum of A is given by the values $\gamma=\pm\frac{1}{2}\pm\sqrt{\mu^2+(\frac{n-1}{2}-p)^2}$ for μ^2 covering the spectrum of $\Delta_{\mathbb{S}^n}$ acting on the coclosed p-forms.

Now the spectrum for the standard sphere has been calculated in [GM75] and as a consequence one has $\mu^2\geq(n-p)(p+1)$ on coclosed p-forms, unless $p=0$ for which we have in fact $\mu^2\geq(n-p)(p+1)$ on coexact p-forms (*ie.* non constant functions). As a consequence

$$\mu^2+(\frac{n-1}{2}-p)^2\geq(n-p)(p+1)+(\frac{n+1}{2}-(p+1))^2=(\frac{n+1}{2})^2$$

and then

$$|\gamma|\geq\frac{n}{2}. \quad (4.7)$$

For $p=0$, the eigenvalues of A corresponding to the constant function are in fact $\pm\frac{n}{2}$ as we can see with the expression of A, so the minoration (4.7) is allways valid and, in particular, $0\notin\mathrm{Spec}(A)$.

consequence. — The elliptic operator $A(A+1)$ is non negative (and positive if $n\geq3$). Indeed $A(A+1)=(A+1/2)^2-1/4$ and the values of the eigenvalues of A give the conclusion.

4.3. **Equations satisfied.** — On the cones, $\sigma = (\sigma_1, \sigma_2)$ satisfies the equations

$$\Big(-\partial_r^2 + \frac{1}{r^2}A(A+1)\Big)\sigma_1 = \lambda_\varepsilon \sigma_1 \tag{4.8}$$

$$\Delta_2 U^*\sigma_2 = \varepsilon^2 \lambda_\varepsilon U^* \sigma_2 \tag{4.9}$$

and the compatibility conditions have been given in (4.3) and (4.4):

$$\sigma_2(1) = \varepsilon^{1/2}\sigma_1(\varepsilon), \quad \sigma_2'(1) = \varepsilon^{3/2}\sigma_1'(\varepsilon). \tag{4.10}$$

We decompose σ_1 along a base of eigenvectors of A : $\sigma_1 = \sum \sigma_1^\gamma$ and $A\sigma_1^\gamma = \gamma\sigma_1^\gamma$.

4.4. **Boundary control.** — We know that $\int_\varepsilon^1 |(\partial_r + \frac{A}{r})\sigma_1|^2 \leq \lambda + 1$ for ε small enough.This inequality stays valid for $\xi_1\sigma_1$ with a bigger constant: there exists $\Lambda > 0$ such that for any $\varepsilon > 0$

$$\sum_{\gamma \in \mathrm{Spec}(A)} \int_\varepsilon^1 |\partial_r(\xi_1\sigma_1^\gamma) + \frac{\gamma}{r}(\xi_1\sigma_1^\gamma)|^2 \leq \Lambda.$$

Then, if we remark that $\partial_r \sigma + \frac{\gamma}{r}\sigma = r^{-\gamma}\partial_r(r^\gamma \sigma)$ we can write, for $\gamma < 0 \Rightarrow \gamma \leq -\frac{n}{2}$,

$$(\varepsilon^\gamma \sigma_1^\gamma(\varepsilon))^2 = \Big(\int_\varepsilon^1 \partial_r(r^\gamma \xi_1 \sigma_1^\gamma)\Big)^2 \leq \int_\varepsilon^1 r^{2\gamma} \int_\varepsilon^1 |\partial_r(\xi_1\sigma_1^\gamma) + \frac{\gamma}{r}(\xi_1\sigma_1^\gamma)|^2 \tag{4.11}$$

So $\sigma_1^\gamma(\varepsilon) = O(\varepsilon^{1/2}/\sqrt{|2\gamma+1|})$. This suggests that the limit σ is harmonic on $M_2(1)$ with boundary condition $\Pi_{<0}\sigma_2 = 0$, if $\Pi_{<0}$ denote the spectral projector of A on the total eigenspace of negative eigenvalues. The limit problem appearing here has a boundary condition of Atiyah-Patodi-Singer type [APS75]. Indeed we have

Proposition 4. *There exists a constant C such that the boundary value satisfies, for all $\varepsilon > 0$*

$$\|\Pi_{<0}\Big(\sigma_{1,\varepsilon}(\varepsilon)\Big)\|^2 \leq C\varepsilon.$$

Proof. We know that $q(\xi_1\varphi_{1,\varepsilon}, \varphi_{2,\varepsilon})$ is bounded by Λ, on the other hand the expression of the quadratic form (4.5) can be done with respect to the decomposition along

$\operatorname{Im}\Pi_{>0}$ and $\operatorname{Im}\Pi_{<0}$. Namely:

$$\begin{aligned}
q(\xi_1\varphi_{1,\varepsilon},\varphi_{2,\varepsilon}) &= \int_\varepsilon^1 \left|\Big(\partial_r+\frac{1}{r}A\Big)\Pi_{<0}(\xi_1\sigma_{1,\varepsilon})\right|^2 dr \\
&\qquad + \int_\varepsilon^1 \left|\Big(\partial_r+\frac{1}{r}A\Big)\Pi_{>0}(\xi_1\sigma_{1,\varepsilon})\right|^2 dr + \frac{1}{\varepsilon^2}\int_{M_2(1)} |D_2\varphi_2|^2 \\
&\geq \int_\varepsilon^1 \left|\Big(\partial_r+\frac{1}{r}A\Big)\Pi_{<0}(\xi_1\sigma_{1,\varepsilon})\right|^2 dr \\
&\geq \int_\varepsilon^1 \Big[\,|\Pi_{<0}(\xi_1\sigma_{1,\varepsilon})'|^2 + \frac{1}{r^2}\langle\Pi_{<0}(\xi_1\sigma_{1,\varepsilon}),(A+A^2)\Pi_{<0}(\xi_1\sigma_{1,\varepsilon})\rangle\,\Big]dr \\
&\qquad - \frac{1}{\varepsilon}\big\langle\Pi_{<0}\sigma_1(\varepsilon), A\circ\Pi_{<0}\sigma_1(\varepsilon)\big\rangle \\
&\geq \frac{n}{2\varepsilon}\|\Pi_{<0}\sigma_{1,\varepsilon}(\varepsilon)\|^2
\end{aligned}$$

because $A(A+1)$ is non negative and $-A\circ\Pi_{<0}\geq\frac{n}{2}$. □

4.5. **Limit problem.** — We study here good candidates for the limit Gauß-Bonnet operator. On M_1 the problem is clear, the question here is to identify the boundary conditions on $M_2(1)$.

• On M_1 the natural problem is the Friedrich extension of D_1 on the cone, it is not a real conical singularity and $\Delta_1 = D_1^*\circ D_1$ is the usual Hodge-de Rham operator (we can see with the expression of φ_1 using the Bessel functions, see appendix, that $\sum_\gamma |d_{\varepsilon,\bar\gamma}|^2\varepsilon^{-2\bar\gamma+1}/2\bar\gamma-1$ is bounded so $\lim_{\varepsilon\to0}\sum_\gamma |d_{\varepsilon,\bar\gamma}|^2=0$ and the limit $U\varphi$ has only regular components, *ie.* in terms of $f_{\bar\gamma}(r)$).

• For $n\geq2$ the forms on $M_2(1)$ satisfying $D_2(\varphi)=0$ $\Pi_{<0}\circ U(\varphi)=0$ on the boundary are precisely the L_2 forms in $\operatorname{Ker}(D_2)$ on the large manifold $\widetilde{M_2}$ obtained from $M_2(1)$ by gluing a conic cylinder $[1,\infty[\times\mathbb{S}^n$ with metric dr^2+r^2h, *ie.* the exterior of the sphere in $\mathbb{R}^{n+1}$.

Indeed, these L_2 forms must satisfy $(\partial_r+\frac{1}{r}A)\sigma=0$ or, $\forall\gamma\in\operatorname{Spec}(A)$, $\exists\sigma_0^\gamma\in\ker(A-\gamma)$ such that $\sigma^\gamma=r^{-\gamma}\sigma_0^\gamma\in L_2$ which is possible only for $\gamma>1/2$. This limit problem is of the category *non parabolic at infinity* in the terminology of Carron [C01], see particularly the theorem 2.1 there, then as a consequence of theorem 0.4 of the same paper we know that its kernel is finite dimensional, more precisely it gives:

Proposition 5. *The operator D_2 acting on the forms of $M_2(1)$, with the boundary condition $\Pi_{<0}\circ U=0$, is elliptic in the sens that the H_1 norm of elements of the domain is controled by the norm of the graph. Let's $\mathcal{D}_2$ denote this operator.*

Corollary 6. *The kernel of $\mathcal{D}_2$ is of finite dimension and can be identify with a subspace of the total space $\sum_p H^p(M_2(1))$ of absolute cohomology.*

We shall see in Corollary 15 below that this kernel is in fact the total space $\sum_p H^p(M_2)$.

Proof. We show that there exists a constant $C > 0$ such that for each $\varphi \in H^1(\Lambda T^* M_2(1))$ satisfying $\Pi_{<0} \circ U(\varphi) = 0$, then

$$\|\varphi\|_{H^1} \leq C(\|\varphi\|_{L_2} + \|D_2(\varphi)\|_{L_2}).$$

Thus $\mathcal{D}_2$ is closable.

Denote, for such a φ, by $\tilde{\varphi}$ its harmonic prolongation on $\widetilde{M}_2$. Then $\tilde{\varphi}$ is in the domain of the Dirac operator on $\widetilde{M}_2$ which is elliptic, it means that for each smooth function f with compact support there exists a constant $C_f > 0$ such that

$$\forall \psi \in \mathrm{dom}(D_2) \quad \|f.\psi\|_{H^1} \leq C_f(\|\psi\|_{L_2} + \|D_2(\psi)\|_{L_2})$$

(it is the fact that an operator 'non parabolic at infinity' is continue from its domain to H^1_{loc}, Theorem 1.2 of Carron)

If we apply this inequality for some $f = 1$ on $M_2(1)$ and $\psi = \tilde{\varphi}$ we obtain in particular that

$$\|\varphi\|_{H^1(M_2(1))} \leq C(\|\tilde{\varphi}\|_{L_2} + \|D_2(\tilde{\varphi})\|_{L_2})$$

with $C = C_f$. We remark first that

$$\|D_2(\tilde{\varphi})\|_{L_2(\widetilde{M}_2)} = \|D_2(\varphi)\|_{L_2(M_2(1))}.$$

Now we can write, by the use of cut-off functions, $\varphi = \varphi_0 + \bar{\varphi}$ with φ_0 null near the boundary and $\bar{\varphi}$ supported in $1/2 \leq r \leq 1$. Then $\tilde{\varphi}_0 = 0$ so, for the control of $\|\tilde{\varphi}\|_{L_2}$, we can suppose that $\varphi = \bar{\varphi}$. We write $U\varphi = \sigma$ and $\sigma = \sum_\gamma \sigma^\gamma$ on the eigenspaces of A. We have

$$\|\tilde{\varphi}\|^2_{L_2(\mathbb{R}^m - B(0,1))} = \sum_{\gamma > 0} \frac{1}{2\gamma - 1} |\sigma^\gamma(1)|^2,$$

now $\gamma \geq 1$ and $\sigma^\gamma(1/2) = 0$, so one has $\sigma^\gamma(1) = \int_{1/2}^1 \partial_r(r^\gamma \sigma^\gamma)$ and by Cauchy-Schwarz inequality

$$|\sigma^\gamma(1)|^2 \leq \int_{1/2}^1 (r^{-\gamma} \partial_r(r^\gamma \sigma^\gamma))^2 \int_{1/2}^1 r^{2\gamma}$$

or

$$|\sigma^\gamma(1)|^2 \leq \|(\partial_r + \frac{1}{r} A)(\sigma^\gamma)\|^2 \frac{1}{2\gamma + 1}$$

as a consequence

$$\sum_{\gamma > 0} \frac{1}{2\gamma - 1} |\sigma^\gamma(1)|^2 \leq \sum_{\gamma > 0} \|(\partial_r + \frac{1}{r} A)(\sigma^\gamma)\|^2 \frac{1}{4\gamma^2 - 1} \leq \|D_2(\varphi)\|^2$$

then, changing the constant, we have also

$$\|\varphi\|_{H^1(M_2(1))} \leq C(\|\varphi\|_{L_2(M_2(1)} + \|D_2(\varphi)\|_{L_2(M_2(1))}.$$

$\square$

alternative proof of the proposition, in the spirit of [APS75]. — To study this boundary condition it is better to write again the p-form near the boundary as $\varphi_2 = dr \wedge r^{-(n/2-p+1)}\beta_2 + r^{-(n/2-p)}\alpha_2$ with, as before, $U(\varphi_2) = \sigma_2 = (\beta_2, \alpha_2)$. On the cone $r \in [1/2, 1]$, $UD_2U^* = \partial_r + \frac{1}{r}A$ and we can construct, as in [APS75] a parametrix of D_2 by gluing an interior parametrix with one constructed on the 'long' cone $r \in]0, 1]$ as follows :

Given a form ψ on $M_2(1)$, if we look for a form φ such that $D_2\varphi = \psi$, we write ψ as the sum of two terms, the first one with support in the neighborood of the boundary and the second one nul near the boundary. On the second term we apply an interior parametrix Q_0 of the elliptic operator D_2. Let's now supposes that φ is supported in the cone $r \in [1/2, 1]$. We decompose $U\psi$ along the eigenspaces of $A : U\psi = \sum_\gamma \psi^\gamma$ and if also $U\varphi = \sum_\gamma \varphi^\gamma$, then φ^γ must satisfy

$$\partial_r\varphi^\gamma + \frac{\gamma}{r}\varphi^\gamma = r^{-\gamma}\partial_r(r^\gamma\varphi^\gamma) = \psi^\gamma.$$

We take the solution

$$\varphi^\gamma = r^{-\gamma}\int_1^r \rho^\gamma\psi^\gamma(\rho)d\rho \text{ if } \gamma < 0 \tag{4.12}$$

$$\varphi^\gamma = r^{-\gamma}\int_0^r \rho^\gamma\psi^\gamma(\rho)d\rho \text{ if } \gamma > 0 \tag{4.13}$$

Thus $\gamma < 0 \Rightarrow \varphi^\gamma(1) = 0$. It is now easy to verify that $\mathcal{D}_2$ satisfies the property (SE) of [L97] p. 54 (with $\rho(x) = \sqrt{x}$).

This fact and the vacuity of $\mathrm{Spec}(A)\cap]-1, +1[$ assure the construction of the parametrix on the cone, see [L97] and also [BS88] who make this construction. In fact the parametrix on the cone gives only H^1 regularity with weight function, but we will cut the singular point for $M_2(1)$, these results are in [L97] Proposition 1.3.12 and following.

4.6. **Boundedness.** — Recall that $A(A+1)$ is non negative.

Proposition 7. *Let χ be a cut-off function supported in $[3/4, 1[$ equal to 1 on $[7/8, 1[$ and $\sigma_{2,\varepsilon} = U(\varphi_{2,\varepsilon})$. The family $\psi_{2,\varepsilon} = \varphi_{2,\varepsilon} - U^*\Big(\Pi_{<0}(\chi\sigma_{2,\varepsilon})\Big)$ belongs to the domain of $\mathcal{D}_2$, is bounded in $H^1(M_2)$ and satisfies $\lim_{\varepsilon\to 0}\|\psi_{2,\varepsilon} - \varphi_{2,\varepsilon}\| = 0$ and*

$$\lim_{\varepsilon\to 0}\|D_2(\psi_{2,\varepsilon} - \varphi_{2,\varepsilon})\| = O(\sqrt{\varepsilon}) \tag{4.14}$$

As a consequence of this result, there exists a subsequence of $\varphi_{2,\varepsilon}$, which converge in L_2 to an harmonic form satisfying the boundary conditions of $\mathcal{D}_2$.

Proof. We write in the following $\sigma_{2,\varepsilon} = \sigma_2$. It is clear that $\psi_{2,\varepsilon}$ belongs to the domain of $\mathcal{D}_2$, and is a bounded family for the operator norm. Thus, by ellipticity it is also a bounded family in $H^1(M_2)$. Now

$$\|\psi_{2,\varepsilon} - \varphi_{2,\varepsilon}\|^2 \le \int_{3/4}^1 |\Pi_{<0}\sigma_2(r)|^2 dr$$

but as a consequence of (4.6)

$$|\Pi_{<0}\sigma_2(r)|^2 = 2\int_{1/2}^{r} \langle \Pi_{<0}\sigma_2'(t), \Pi_{<0}\sigma_2(t)\rangle dt + |\Pi_{<0}\sigma_2(1/2)|^2 \leq 2\varepsilon\Lambda + \varepsilon^2 \frac{2}{n}\Lambda \quad (4.15)$$

using the inequality of Cauchy-Schwarz, the fact that the L_2-norm of φ_ε is 1 and that $(-A \circ \Pi_{<0}) \geq \frac{n}{2}$. For the second estimate:

$$D_2(\varphi_{2,\varepsilon} - \psi_{2,\varepsilon}) = D_2U^*\Big(\Pi_{<0}(\chi\sigma_{2,\varepsilon})\Big) = \chi' U^* \Pi_{<0}(\sigma_{2,\varepsilon}) + \chi D_2 U^* \Pi_{<0}(\sigma_{2,\varepsilon})$$

and the norm of the first term is controled by $\int_{3/4}^{1} |\Pi_{<0}\sigma_2(r)|^2 dr$ which is $O(\varepsilon)$ by the estimate(4.15) and the norm of the second term by $\|D_2(\varphi_2)\|$ which is $O(\varepsilon)$ because $q_\varepsilon(\varphi_\varepsilon)$ is uniformly bounded (remark that D_2 preserves the orthogonal decomposition following $\Pi_{<0}$ and $\Pi_{>0}$ on the cone). □

Corollary 8. *The family $\Pi_{>0}\sigma_2(1)$ is bounded in $H^{1/2}(\mathbb{S}^n)$ as the boundary value of $\psi_{2,\varepsilon}$.*

We now define a better prolongation of $\Pi_{>0}\sigma_2(1)$ on $M_1(\varepsilon)$. More generally let

$$P_\varepsilon : \Pi_{>0}\Big(H^{1/2}(\mathbb{S}^n)\Big) \to H^1(\mathcal{C}_{\varepsilon,1}) \quad (4.16)$$

$$\sigma = \sum_{\gamma \in \mathrm{Spec}(A), \gamma>0} \sigma_\gamma \mapsto P_\varepsilon(\sigma) = \sum_{\gamma \in \mathrm{Spec}(A), \gamma>0} \varepsilon^{\gamma-1/2} r^{-\gamma} \sigma_\gamma. \quad (4.17)$$

We remark that there exists a constant C such that

$$\|P_\varepsilon(\sigma)\|^2_{L_2(M_1(\varepsilon))} \leq C \sum |\sigma_\gamma|^2 = C \|\Pi_{>0}\sigma_2(1)\|^2_{L_2(\mathbb{S}^n)} \quad (4.18)$$

and also that, if $\psi_2 \in \mathrm{Dom}\mathcal{D}_2$ and with the same cut-off function ξ_1, which has value 1 for $0 \leq r \leq 1/2$ and 0 for $r \geq 1$, then $\Big(\xi_1 P_\varepsilon(\psi_{2|\mathbb{S}^n}), \psi_2\Big)$ defines through the isometries U an element of $H^1(M_\varepsilon)$. Let

$$\tilde{\psi}_1 := \xi_1 P_\varepsilon(\psi_{2|\mathbb{S}^n}).$$

We now decompose $\varphi_{1,\varepsilon}$ as follows. Let

$$\xi_1 \varphi_{1,\varepsilon} = \xi_1(\varphi_\varepsilon^+ + \varphi_\varepsilon^-)$$

according to the decomposition of σ_1 along the positive or negative spectrum of A on the cone. Then $\tilde{\psi}_1$ and φ_ε^+ have the same values on the boundary so the difference $\xi_1\varphi_\varepsilon^+ - \tilde{\psi}_1$ can be viewed in $H^1(M_1)$ by a prolongation by 0 on the ball, while the boundary value of φ_ε^- is small. We introduce for this term the cut-off function taken in [ACP07]

$$\xi_\varepsilon(r) = \begin{cases} 1 & \text{if } r \geq 2\sqrt{\varepsilon}, \\ \dfrac{\log(2\varepsilon) - \log r}{\log(\sqrt{\varepsilon})} & \text{if } r \in [2\varepsilon, 2\sqrt{\varepsilon}], \\ 0 & \text{if } r \leq 2\varepsilon. \end{cases}$$

Lemma 9. $\lim_{\varepsilon\to 0} \|(1-\xi_\varepsilon)\xi_1\varphi_\varepsilon^-\|_{L_2} = 0.$

This is a consequence of the estimate of the Proposition 4.

Proposition 10. *The forms* $\psi_{1,\varepsilon} = (1-\xi_1)\varphi_{1,\varepsilon} + (\xi_1\varphi_\varepsilon^+ - \tilde{\psi}_1) + \xi_\varepsilon\xi_1\varphi_\varepsilon^-$ *belong to* $H^1(M_1)$ *and define a bounded family.*

Proof. We will show that each term is bounded. For the first one it is already done in Proposition 1. For the second one, we remark that

$$(\partial_r + \frac{A}{r})(\varphi_\varepsilon^+ - \tilde{\psi}_1) = (\partial_r + \frac{A}{r})(\varphi_\varepsilon^+) + \partial_r(\xi_1)P_\varepsilon(\psi_{2|\mathbb{S}^n}) := f_\varepsilon \tag{4.19}$$

and f_ε is uniformly bounded in $L_2(M_1)$ because of (4.18). This estimate (4.18) shows also that the L_2-norm of $(\varphi_\varepsilon^+ - \tilde{\psi}_1)$ is bounded. Thus the family $(\xi_1\varphi_\varepsilon^+ - \tilde{\psi}_1)$ is bounded for the q-norm in $H^1(M_1)$ which is equivalent to the H^1-norm.
For the third one we use the estimate due to the expression of the quadratic form. Expriming that $\int_{\mathcal{C}_{r,1}} |D_1(\xi_1\varphi^-)|^2$ is bounded by Λ gives that

$$\frac{1}{r}\langle\sigma_1^-(r), |A|\sigma_1^-(r)\rangle \leq \Lambda \tag{4.20}$$

by the same argument as used for the Proposition 4. Now

$$\|D_1(\xi_\varepsilon\xi_1\varphi_\varepsilon^-)\| \leq \|\xi_\varepsilon D_1(\xi_1\varphi_\varepsilon^-)\| + \||d\xi_\varepsilon|\xi_1\varphi_\varepsilon^-\| \leq \|D_1(\xi_1\varphi_\varepsilon^-)\| + \||d\xi_\varepsilon|\xi_1\varphi_\varepsilon^-\|$$

the first term is bounded and, with $|A| \geq \frac{n}{2}$ and the estimate (4.20), we have

$$\begin{aligned}\||d\xi_\varepsilon|\xi_1\varphi_\varepsilon^-\|^2 &\leq \frac{8\Lambda}{n\log^2\varepsilon}\int_\varepsilon^{\sqrt{\varepsilon}}\frac{dr}{r}\\ &\leq \frac{4\Lambda}{n|\log\varepsilon|}.\end{aligned}$$

This complete the proof. □

In fact the decomposition used here is almost orthogonal:

Lemma 11.

$$< (\varphi_\varepsilon^+ - \tilde{\psi}_1), \tilde{\psi}_1 >== O(\sqrt{\varepsilon}).$$

proof of lemma 11. — If we decompose the terms under the eigenspaces of A we see that only the positive eigenvalues are involved and, with $f_\varepsilon = \sum_{\gamma>0} f^\gamma$ and $(\varphi_\varepsilon^+ - \tilde{\psi}_1) = \sum_{\gamma>0}\varphi_0^\gamma$, the equation (4.19) and the fact that $(\varphi_\varepsilon^+ - \tilde{\psi}_1)(\varepsilon) = 0$ give

$$\varphi_0^\gamma(r) = r^{-\gamma}\int_\varepsilon^r \rho^\gamma f^\gamma(\rho)d\rho.$$

Then for each positive eigenvalue γ of A

$$
\begin{aligned}
< (\varphi_0^\gamma, \tilde{\psi}_1^\gamma > &= \varepsilon^{\gamma-1/2} \int_\varepsilon^1 r^{-2\gamma} \int_\varepsilon^r \rho^\gamma < \sigma_\gamma, f^\gamma(\rho) >_{L_2(\mathbb{S}^n)} d\rho \\
&= \varepsilon^{\gamma-1/2} \int_\varepsilon^1 \frac{r^{-2\gamma+1}}{2\gamma-1} r^\gamma < \sigma_\gamma, f^\gamma(r) >_{L_2(\mathbb{S}^n)} dr + \\
&\qquad\qquad \frac{\varepsilon^{\gamma-1/2}}{2\gamma-1} \int_\varepsilon^1 \rho^\gamma < \sigma_\gamma, f^\gamma(\rho) >_{L_2(\mathbb{S}^n)} d\rho \\
&\leq \varepsilon^{\gamma-1/2} \int_\varepsilon^1 \frac{r^{-\gamma+1}}{2\gamma-1} < \sigma_\gamma, f^\gamma(r) >_{L_2(\mathbb{S}^n)} dr + \\
&\qquad\qquad \frac{\varepsilon^{\gamma-1/2}}{(2\gamma-1)\sqrt{2\gamma+1}} \|\sigma_\gamma\| \|f^\gamma\|_{L_2(\mathcal{C}_{\varepsilon,1})} \\
&\leq C\varepsilon^{\gamma-1/2} \|\sigma_\gamma\| \frac{\varepsilon^{(-2\gamma+3)/2}}{(2\gamma-1)(2\gamma-1)\sqrt{2\gamma-3}} \|f^\gamma\|_{L_2(\mathcal{C}_{\varepsilon,1})} + \\
&\qquad\qquad \frac{\varepsilon^{\gamma-1/2}}{(2\gamma-1)\sqrt{2\gamma+1}} \|\sigma_\gamma\| \|f^\gamma\|_{L_2(\mathcal{C}_{\varepsilon,1})} \\
&\leq C\sqrt{\varepsilon} \|\sigma_\gamma\| \|f^\gamma\|_{L_2(\mathcal{C}_{\varepsilon,1})}.
\end{aligned}
$$

This estimate gives the lemma.

Remark: For $\gamma > 1$, and so for $n > 2$, this estimate is better.

5. PROOF OF THEOREM B

Lemma 12. *If $\lambda \neq 0$, then $\lambda_\varepsilon \neq 0$ for all ε and*

$$\lim_{\varepsilon \to 0} (L_2) \tilde{\psi}_{1,\varepsilon} = 0$$

and also

$$\lim_{\varepsilon \to 0} (L_2) \psi_{2,\varepsilon} = 0$$

as well as in q-norm.

Proof. We know, by the Proposition 1, that there is a universal lower bound for positive eigenvalues on $M(\varepsilon)$, so if $\lambda = \lim \lambda_\varepsilon$ is positive, it means that all the λ_ε are also positive! We know that $\psi_{2,\varepsilon}$ is in the domain of $\mathcal{D}_2$, we decompose

$$\psi_{2,\varepsilon} = \psi_{2,\varepsilon}^0 + \bar{\psi}_{2,\varepsilon}$$

along $\operatorname{Ker} \mathcal{D}_2$ and its orthogonal. Each part is bounded in $H^1(M_2(1))$ and can be prolongated on the cone using P_ε.

Let $\tilde{\psi}_{1,\varepsilon}^0 = \xi_1 P_\varepsilon(\psi_{2|\mathbb{S}^n}^0)$, $\bar{\psi}_{1,\varepsilon} = \xi_1 P_\varepsilon(\bar{\psi}_{2|\mathbb{S}^n})$ and

$$\psi_\varepsilon = (\tilde{\psi}_{1,\varepsilon}^0 + \bar{\psi}_{1,\varepsilon}, \psi_{2,\varepsilon}).$$

Now

$$\psi_\varepsilon^0 = (\tilde{\psi}_{1,\varepsilon}^0, \psi_{2,\varepsilon}^0) \in \operatorname{dom}(q).$$

The L_2-norm of ψ_ε^0 is bounded and

$$\begin{aligned} q(\psi_\varepsilon^0) &= \int_\varepsilon^1 |\xi_1'(r) P_\varepsilon(\sigma_2^0)|^2 \\ &\leq C \int_{1/2}^1 |P_\varepsilon(\psi^0_{2|\mathbb{S}^n})|^2 \\ &\leq O(\varepsilon^{n-1}) \end{aligned}$$

due to the expression of P_ε the fact that $\operatorname{spec}(|A|) \geq \frac{n}{2}$ and the uniform boundedness of P_ε. Because $n \geq 2$ and Proposition 1 is true, we conclude that the distance of ψ_ε^0 to $\operatorname{Ker} \Delta_\varepsilon$ is $O(\varepsilon)$. But we know that $\lambda_\varepsilon \neq 0$, so φ_ε is orthogonal to $\operatorname{Ker} \Delta_\varepsilon$ and, with the previous result

$$< \varphi_\varepsilon, \psi_\varepsilon^0 >= O(\sqrt{\varepsilon}).$$

On the other hand we have that

$$\int |\mathcal{D}_2 \bar{\psi}_{2,\varepsilon}|^2 = O(\varepsilon) \Rightarrow \|\bar{\psi}_{2,\varepsilon}\|_{L_2(M_2(1))} = O(\sqrt{\varepsilon})$$

and finally $\|\bar{\psi}_{2,\varepsilon}\|_{H^1(M_2(1))} = O(\sqrt{\varepsilon})$ by ellipticity so $\|\bar{\psi}_{1,\varepsilon}\|_{L_2(M_1(\varepsilon))} = O(\sqrt{\varepsilon})$ by uniform continuity of P_ε. and we have also

$$< \varphi_\varepsilon, \psi_\varepsilon >= O(\sqrt{\varepsilon}).$$

Now we use Proposition 7 and Lemma 11, the conclusion is

$$\lim_{\varepsilon \to 0} \|\tilde{\psi}_{1,\varepsilon}\|^2 + \|\psi_{2,\varepsilon}\|^2 = 0.$$

□

As a consequence of this result and Proposition 7, we obtain

Corollary 13. $\lim_{\varepsilon \to 0} (L_2) \varphi_{2,\varepsilon} = 0$.

Recall now that $\psi_{1,\varepsilon} = \varphi_{1,\varepsilon} - \tilde{\psi}_{1,\varepsilon} - (1 - \xi_\varepsilon)\xi_1 \varphi_\varepsilon^-$ and that we know, by the last Lemma and Lemma 9, that the two last terms converge to 0.

Corollary 14. *We can extract from $\psi_{1,\varepsilon}$ a subsequence which converge in L_2 and weakly in H^1, and any such subsequence defines at the limit a form $\varphi \in H^1(M_1)$ such that*

$$\|\varphi\|_{L_2} = 1 \textit{ and } \Delta\varphi = \lambda\varphi \textit{ weakly.}$$

6. Proof of theorem C

6.1. **multiplicity of 0.** The dimension of the kernel of Δ_ε is given by the cohomology of M which can be calculated with the Mayer-Vietoris sequence associated to the covering U_1, U_2 introduced at the beginning, see Proposition 1. If we remember also that $H^p(M_j - B, \mathbb{R}) \sim H^p(M_j, \mathbb{R})$ for $p < m$, we obtain that $H^p(M, \mathbb{R}) \sim H^p(M_1, \mathbb{R}) \oplus H^p(M_2, \mathbb{R})$ for $1 \leq p \leq (m-1)$ while $H^{0,m}(M, \mathbb{R}) \sim H^{0,m}(M_1, \mathbb{R}) \sim H^{0,m}(M_2, \mathbb{R})$.

The transplantation of the harmonic forms of M_1 in M has been described in [AC93]. With the previous calculation, we have good candidates for transplantation of the cohomology of M_2: for each $\sigma_2 \in \operatorname{Ker} \mathcal{D}_2$ with L_2-norm equal to 1, let

$$\tilde{\psi}_\varepsilon = (\tilde{\psi}_1, \psi_2) = U^*\Big(\xi_1 P_\varepsilon(\sigma_{2|\mathbb{S}^n}), \sigma_2\Big).$$

Now let $\varphi_\varepsilon \in \operatorname{Ker} \Delta_\varepsilon$. We apply to φ_ε the preceding estimates: there exists a subsequence which gives at the limit $\psi_1 \in \operatorname{Ker} \Delta_1$ and $\psi_2 \in \operatorname{Ker} \mathcal{D}_2$; and only one of these two terms can be zero. The conclusion is that all the harmonic forms of M_ε can be approched by forms like $\tilde{\psi}_\varepsilon$ or $\chi_\varepsilon \varphi_1$, with $\varphi_1 \in \operatorname{Ker} \Delta_1$. As a consequence on has

Corollary 15. *For $1 \leq p \leq (m-1)$ the two spaces $H^p(M_2, \mathbb{R})$ and $\operatorname{Ker} \mathcal{D}_2$ are isomorphic.*

6.2. **convergence of the positive spectrum.** The proof is made by induction. We show first that $\lim \lambda_1(\varepsilon) = \lambda_1$:

Proof. We know by the Proposition A that $\limsup \lambda_1(\varepsilon) \leq \lambda_1$ and by Proposition B that $\liminf \lambda_1(\varepsilon)$ is in the positive spectrum of Δ_1, and as a consequence $\liminf \lambda_1(\varepsilon) \geq \lambda_1$. □

Now suppose that for all j, $1 \leq j \leq k$ one has $\lim \lambda_j(\varepsilon) = \lambda_j$, we have to show that $\lim \lambda_{k+1}(\varepsilon) = \lambda_{k+1}$.

Proof. We know by Proposition A that $\limsup \lambda_{k+1}(\varepsilon) \leq \lambda_{k+1}$; let $\varphi_\varepsilon^{(1)}, \dots, \varphi_\varepsilon^{(k+1)}$ be an orthonormal family of eigenforms on $M(\varepsilon)$:

$$\Delta_\varepsilon \varphi_\varepsilon^{(j)} = \lambda_j(\varepsilon) \varphi_\varepsilon^{(j)}$$

and choose a sequence $\varepsilon_l \to 0$ such that

$$\lim_{l \to \infty} \lambda_{k+1}(\varepsilon_l) = \liminf \lambda_{k+1}(\varepsilon).$$

We apply to each $\varphi_\varepsilon^{(j)}$ the same decomposition as in Proposition 10, this gives a family $\psi_\varepsilon^{(1)}, \dots, \psi_\varepsilon^{(k+1)}$ bounded in $H^1(M_1)$ and such that for each indice j

$$\lim_{\varepsilon \to 0} \|\varphi_{1,\varepsilon}^{(j)} - \psi_\varepsilon^{(j)}\| = 0$$

while, as in Corollary 13

$$\lim_{\varepsilon \to 0} (L_2) \varphi_{2,\varepsilon}^{(j)} = 0.$$

So, by extraction of a subsequence,we can suppose that $\psi_{\varepsilon_l}^{(1)}, \dots, \psi_{\varepsilon_l}^{(k+1)}$ converge in $L_2(M_1)$ and weakly in $H^1(M_1)$, the limit $\varphi^{(1)}, \dots, \varphi^{(k+1)}$ is orthonormal and satisfies

$$\forall j,\ 1 \leq j \leq k \Delta_1 \varphi^{(j)} = \lambda_j \varphi^{(j)} \text{ and } \Delta_1 \varphi^{(k+1)} = \liminf \lambda_{k+1}(\varepsilon) \varphi^{(k+1)}.$$

This shows that $\liminf \lambda_{k+1}(\varepsilon) \geq \lambda_{k+1}$ and finishes the proof. □

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